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C. Bernardini: THE Z-DISTRIBUTION OF AN ELECTRON BEAM IN A STORAGE RING.

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1. - Introduction

The luminosity L of a $(e^+ - e^-)$ storage ring can
be defined as

$$L = \iint dx\, dz\; n_+(x,z)\, n_-(x,z)$$

where $n_+(x, z)\, dx\, dz$ is the number of $e^{\pm}$ having radial
coordinate x and vertical coordinate z. The rate of events,
having a cross section σ is then

$$\dot{n} = \frac{f}{k} L \sigma$$

at each interaction region in the ring. Here f is the rota-
tion frequency and k the RF harmonic. If $N_\pm$ is the total
number of stored $e^{\pm}$, write

$$n_\pm = N_\pm\, P_\pm(x,z)$$

so that the usually defined "cross section of the beam" s
can be introduced as follows

$$\frac{1}{s} = \iint dx\, dz\; P_+(x,z)\, P_-(x,z)$$

Then L becomes simply

$$\angle = \frac{N_+ N_-}{S}$$

We will assume in the following that:

    a) there is no x-z coupling

    b) e^+ and e^- follow the same path with opposite veloci
       ties.

This means that

$$p_+(x,z) = p_-(x,z) = p_1(x) p_2(z)$$

we define

$$\frac{1}{a_r} = \int_{-\infty}^{+\infty} p_r^2(y) \, dy \qquad\qquad (r = 1, 2)$$

so that

$$S' = a_1 a_2$$

We want to give here a general formula for $p_2(z)$ and an estimate of a_2 valid in a broad range of practical conditions.

2. - The z-distribution

Let us abbreviate $p(z)$ for $p_2(z)$. This is the steady distribution the electrons arrive at under the action of damping in competition with two random forces:

    a) synchrotron radiation (quantum effects)

    b) gas scattering

The synchrotron radiation fluctuations produce a gaussian-like distribution we want not to recalculate here[1]. The gas scattering is, for a storage ring, in a somewhat special situation as compared with conventional machines. In fact, because of the extremely low vacuum and of the very efficient radiation damping, single scattering is the main effect of the gas. We want now to calculate what the effect of the gas can be and to show that the radiation ef-

fect largely prevails in the case of actual electron stora-
ge rings.

Let us write the equations of motion for the z-mo
de as follows:

$$v = \frac{dz}{dt}$$

$$-\beta v - \omega^2 z = \frac{dv}{dt}$$

where β is the radiation damping constant and ω the verti
cal betatron frequency.

The diffusion equation for P(z, v), the probabili
ty density in z-v space, is

$$-v\frac{\partial P}{\partial z} + \omega^2 z \frac{\partial P}{\partial v} + \beta \frac{\partial}{\partial v} v P - kP + k\int_{-\infty}^{+\infty} \psi(w) P(z, v-w) dw = 0$$

This is the equation for the steady distribution; here k is
the inverse mean free time for collisions and $\psi(w)$ is the
distribution of scattering angles in the vertical plane, nor
malized so that

$$\int_{-\infty}^{+\infty} \psi(w) dw = 1$$

w = c x (projected scattering angle)

Let us introduce the double Fourier transform

$$G(\lambda, \mu) = \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dv \, e^{-i(\lambda z + \mu v)} P(z, v)$$

and define

$$Q(\mu) = \int_{-\infty}^{+\infty} \psi(w) e^{-iw\mu} dw$$

Then the diffusion equation changes into

$$\lambda \frac{\partial G}{\partial \mu} - \omega^2 \mu \frac{\partial G}{\partial \lambda} - \beta \mu \frac{\partial G}{\partial \mu} - k[1 - Q(\mu)] G = 0$$

Since by obvious simmetry reasons

$$\left(\frac{\partial G}{\partial \lambda}\right)_{\lambda=0} \sim \int_{-\infty}^{+\infty} dz \, P(z, v) = 0$$

we can immediately calculate $G(o, \mu) = g(\mu)$ from the equation

(1)
$$\beta \mu \frac{dg}{d\mu} + k_o (1 - Q(\mu)) g = 0$$

$g(o) = 1$ because of normalization of $P(z, v)$ to unity when integrated over the whole z-v space.

The solution of eq. (1) is

$$g(\mu) = exp - \frac{k_o}{\beta} \int_0^{\mu} \frac{1 - Q(\bar{\mu})}{\bar{\mu}} d\bar{\mu}$$

Assuming that

$$\mathcal{V}(\omega) = \frac{1}{2} \frac{\omega_s^2}{(\omega^2 + \omega_s^2)^{3/2}}$$

that is, a screened Rutherford cross section with screening angle ω_s/c it follows that

$$g(\mu) = \left(\frac{2 e^{-\gamma}}{|\mu \omega_s|} \right)^{\frac{k_o}{\beta}} e^{-\frac{k_o}{\beta} k_o (|\mu \omega_s|)}$$

where $\gamma = 0.5772$ (the Euler constant) and k_o is a modified Bessel function of 2nd kind.

Now, since $\omega \gg \beta$, we have with a fairly good approximation that the z-distribution is similar to the v-distribution, that is

$$P(z) = \int_{-\infty}^{+\infty} P(z, v) dv \simeq \frac{\omega}{\pi} \int_0^{+\infty} \cos \mu \omega z \; g(\mu) d\mu$$

It follows that

$$P(z) = \frac{\omega}{\pi \omega_s} I_{k/\beta} \left(\frac{\omega z}{\omega_s} \right) (2 e^{-\gamma})^{\frac{k_o}{\beta}}$$

where

$$I_\nu (x) = \int_0^{\infty} \left(\frac{1}{y} \right)^\nu e^{-\nu k_o(y)} \cos x y \; dy$$

what matters is the folding of this distribution with the gaussian distribution due to radiation effects, that is

$$P_t (z) = \int_{-\infty}^{+\infty} P(z - \bar{z}) \; P_{rad} (\bar{z}) \; d\bar{z}$$

Let us write

$$P_{rad}(z)\, dz = \frac{\alpha}{\sqrt{\pi}}\, e^{-\alpha^2 x^2}\, dx$$

where $x\, \omega_s = \omega z$

Then

$$P_t(z) = \frac{\omega}{\omega_s}\, M_\nu(x;\alpha)$$

where $\nu = \frac{k}{\beta}$ and

$$M_\nu(x,\alpha) = \frac{1}{\pi}(2\, e^{-\delta})^\nu \int_0^{+\infty} \left(\frac{2}{y}\right)^\nu e^{-\nu K_0(y)} \cos xy\, e^{-\frac{1}{4}\frac{y^2}{\alpha^2}}\, dy$$

The central density is

$$P_t(0) = \frac{\omega}{\omega_s}\, M_\nu(0;\alpha)$$

The relevant parameter for the luminosity is given by

$$(2) \qquad \frac{1}{a_z} = \int_{-\infty}^{+\infty} P_t^2(z)\, dz = \frac{\omega}{\omega_s}\, M_{2\nu}\left(0, \frac{\alpha}{\sqrt{2}}\right)$$

3. - Numerical considerations

The damping constant β is given by

$$\beta = \frac{W}{E}$$

where W is the radiated power and E the energy of the electrons. In the case of AdA[2]

$$\beta = 15.6\left(\frac{E_{MeV}}{100}\right)^3 sec^{-1}$$

The projected differential scattering cross section has been written

$$d\sigma = \frac{4\pi r_e^2\, z^2}{\gamma^2 \theta_s^2}\, \mathcal{U}(w)\, dw = \sigma_0\, \mathcal{U}(w)\, dw$$

where $\gamma mc^2 = E$, r_e is the classical electron radius, θ_s is the screening angle and Z the atomic number of the residual gas.

Assuming

$$\theta_s = \frac{\gamma_s}{\gamma}$$

where

$$\gamma_s \simeq 0.01 \, Z^{1/3}$$

the total cross section σ_s does not depend on the energy and is

$$\sigma_o = 10^{-20} \, Z^{4/3} \, cm^2$$

If N is the number of molecules per unit volume in the residual gas,

$$N = 3.2 \times 10^{16} \, P_{mmHg} \qquad cc^{-1}$$

then

$$\nu = \frac{k}{\beta} = \frac{N\sigma_o c}{\beta} = 6.4 \times 10^5 \, Z^{4/3} \left(\frac{100}{E_{MeV}}\right)^3 P_{mmHg}$$

In the case of a residual gas not very different from air, the effective Z is about 7, so that

$$\nu \simeq 10^7 \left(\frac{100}{E}\right)^3 P_{mmHg} \qquad ,$$

ν is just the number of damping times in a mean free time. Thus, for instance, when $P = 10^{-9}$ mmHg the scattering is mainly single at every energy > 20 MeV (for AdA).

Coming back to formula (2), § 2, it can be seen that when $\nu \ll 1$ the value of a_2 is essentially the one derived by the radiation effect; to the first order in ν

$$\frac{1}{a_2} = \frac{\alpha \omega}{\sqrt{2\pi} \, \omega_s} \left(1 - 2\nu \log \sqrt{2} \, \alpha\right)$$

or, in a more familiar form

$$a_2 \simeq 2\sqrt{\pi} \, \Delta z \left(1 + 2\nu \log \frac{\hbar \theta_s}{\Delta z}\right)$$

where

$$\Delta Z^2 = \int_{-\infty}^{+\infty} Z^2 P_{rad}(z)\,dz$$

and $\lambda = \frac{c}{\omega}$ is the betatron wavelength divided by 2π. According to Kolomensky and Lebedev[1] the value of ΔZ^2 is given by

$$\Delta Z^2 = \frac{13\sqrt{3}}{96} \frac{\lambda_c R}{n}$$

where λ_c is $\hbar/mc$, the Compton wavelength of the electron and R is the radius of the principal orbit. In the case of AdA the value of ΔZ is

$$\Delta Z = 4.2 \times 10^{-5} \text{ cm}$$

Moreover, the relevant quantity for a_2 is

$$2\sqrt{\pi}\,\Delta Z = 1.5 \times 10^{-4} \text{ cm} = 1.5\,\mu$$

Thus, for AdA, the correction is of the order $10\,\nu$ since $\lambda \theta_s \simeq 100\,\Delta Z$ at $E = 200$ MeV.

The other extreme case is the conventional one in which the scattering is multiple, or $\nu \gg 1$. In this case one must be careful in handling the formula for $g(\mu)$: to relate the results to well known data, come back to the definition of $Q(\mu)$, § 2, introduce a cut off angle θ_{max} in the cross section at $\theta \gg \theta_s$ and then develop $Q(\mu)$ in power series of μ to the order μ^2. One gets:

$$Q(\mu) \simeq 1 - \frac{1}{2}\mu^2 \omega_s^2 \log \frac{2\theta_{max}}{\theta_s}$$

then

$$-\log g(\mu) = + \frac{1}{4}\nu \omega_s^2 \log \frac{2\theta_{max}}{\theta_s}\mu^2 =$$

$$= \frac{1}{4}\left(\frac{\mu}{\mu_0}\right)^2$$

and eventually

$$P(z) = \frac{\omega \mu_0}{\sqrt{\pi}} e^{-\omega^2 z^2 \mu_0^2}$$

This is the conventional result as it is known from the usual theory of multiple scattering: we do not pro ceed any more since this case is of no interest for storage rings.

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References

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